Entanglement entropy and the F theorem

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Introduction

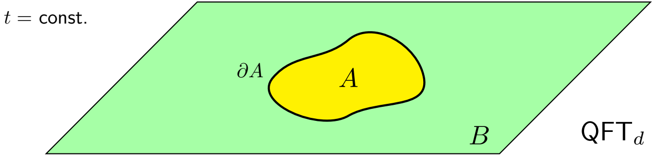
- This talk will be about:
  1. Entanglement entropy
  2. The F theorem for 3d field theories
- Both topics have been of considerable current interest and are intimately connected to each other.

Introduction: Entanglement entropy

- Consider a spatial region A and a density matrix ρ.
- Define ρ_A as the reduced matrix obtained by tracing out all degrees of freedom outside region A.
- The associated von Neumann entropy is the entanglement entropy i.e. $S_A = -\mathrm{Tr}(\rho_A \log \rho_A)$.

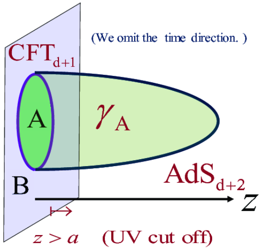
S_A is highly non-trivial to calculate, even in free quantum field theories....

Entangling region



Introduction: Holographic entanglement entropy

Entanglement entropy can be computed geometrically for field theories admitting a gravity dual in one higher dimension.



(Takayanagi)

- Holographic Ryu-Takayanagi (RT) prescription: area of co-dimension two minimal surface homologous to A

$$S_A = \frac{\mathcal{A}}{4G}$$

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Properties of entanglement entropy

- UV divergent: leading divergence scales as the area of ∂A, the boundary of region A.
- Computable (and perhaps observable) in CMT, where lattice acts as UV cutoff.
- Order parameter for phase transitions.

The F theorem

- In a 2d CFT, the anomalous trace of the stress energy tensor is

$$T = c\mathcal{R}$$

  with c the central charge and R the Ricci scalar of the background.
- Zamolodchikov's c-theorem states that c is stationary at a fixed point and decreases monotonically along an RG flow.
- In even dimensions, there are proposals for a theorems.

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F quantity in 3d CFTs

- In a 3d CFT we define the F quantity as

$$F = -\ln Z_{S^3}$$

  and the F theorem states that $F_{UV} \geq F_{IR}$.

More precisely:

- F is conjectured to be stationary at the fixed point, positive in a unitary CFT and to decrease along an RG flow.

(Jafferis, Klebanov, Pufu, Safdi, ...)

Relevance of entanglement entropy

- For a CFT in even dimensions the entanglement entropy in the ground state contains universal terms

$$S_A \sim (-)^{\frac{d}{2}-1} a \log(\frac{R}{\epsilon})$$

where R characterises the scale of the entangling region A, ϵ is a UV cutoff and a is the coefficient appearing in the a theorem.

- In odd dimensions finite terms

$$S_A = 2\pi(-)^{\frac{(d-1)}{2}} a$$

are related to the F quantity (Casini/Huerta/Myers).

Scheme dependence and renormalisation

- Both the partition function on S^3 and the entanglement entropy are UV divergent.
- Implicitly, the F quantity is the renormalized partition function on S^3 - scheme dependence?
- The finite terms in the entanglement entropy look ambiguous: as one shifts the cutoff the finite part changes.

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Holographic entanglement entropy

In this talk we will use holography to gain insight into:

1. The finite contributions to entanglement entropy i.e. renormalization.
2. The relation to the F theorem.

References

- Marika Taylor and William Woodhead

  1. Renormalized entanglement entropy, 1604.06808
  2. The holographic F theorem, 1604.06809

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Outline

- **Renormalized entanglement entropy**
  1. **CFT: minimal surfaces in AdS**
  2. General definition
- The F theorem

Divergent terms in the entanglement entropy

Consider a 3d CFT:

- The entanglement entropy in the vacuum behaves as

$$S_A = c_{-1}\frac{L}{\epsilon} + c_0 + \cdots$$

  where (c_0, c_{-1}) are dimensionless, L is the length of the boundary of the entangling region and $\epsilon \to 0$ is the UV cutoff.
- In a quantum critical CMT system ϵ would be related to the lattice spacing and c_{-1} might be “physical".
- In a continuum QFT, we usually regulate with ϵ and then renormalize.

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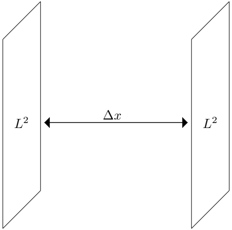
Previous attempts at renormalization

Based on differentiating with respect to parameters:



- For a slab domain in a local QFT, divergences in S must be independent of Δx.
- Therefore

$$S_{\Delta x} \equiv \frac{\partial S}{\partial \Delta x}$$

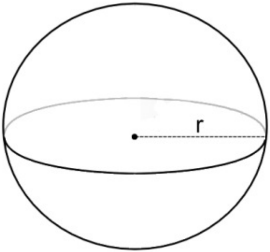
is UV finite.
(e.g. Cardy and Calabrese)

Geometry dependence

- For a spherical region of radius R, divergences in S depend on R.
- For a 3d CFT (disk region) the combination

$$S(R) = \left(R \frac{\partial S}{\partial R} - S \right)$$

is UV finite. (Liu and Mezei)



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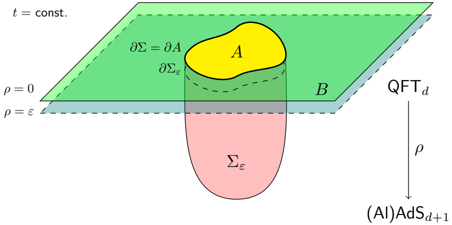
Limitations of this approach

Current interest in dependence of entanglement entropy on shape and theory but:

- No definition for generic shape entangling region.
- $S(R)$ does not generically seem to be stationary at a fixed point (see Jafferis et al).
- Scheme dependence is obscure.

Area renormalization

Turning to holography:



- The natural UV cutoff is $r = \epsilon$.
- One can regulate the area of the minimal surface and define a renormalized area using appropriately covariant counterterms.

Earlier work on renormalized minimal surfaces: (Henningson/Skenderis; Graham/Witten; Gross et al)

Renormalized entanglement entropy

- The Ryu-Takayanagi functional is

$$S = \frac{1}{4G} \int_{\Sigma} d^{d-1}x \sqrt{\gamma}$$

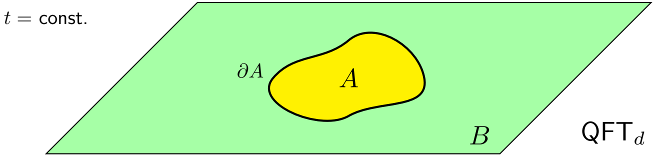
- Use the equations for the minimal surface to expand the surface area asymptotically near the conformal boundary and regulate divergences.
- Covariant counterterms are

$$S_{\rm ct} \sim \int_{\partial\Sigma} d^{d-2}x \sqrt{h} \mathcal{L}(\mathcal{R}, \mathcal{K})$$

where $\mathcal{K}$ is the extrinsic curvature of $\partial\Sigma$ into $r = \epsilon$.

Extrinsic curvature of entangling region

- Complementarity: for *A* and *B* to have the same renormalized entanglement entropy, we can include only terms which are even in the extrinsic curvature.



Results for 3d CFT

- The renormalized EE for an entangling surface in AdS_4 is

$$S_{\rm ren} = \frac{1}{4G}\int_{\Sigma} d^2x\sqrt{\gamma} - \frac{1}{4G}\int_{\partial\Sigma} dx\sqrt{h}(1 - c_s\mathcal{K})$$

  with $\partial\sigma$ the boundary of the minimal surface.
- Here $\mathcal{K}$ is the extrinsic curvature of the bounding curve (into the cutoff surface).
- Complementarity implies that $c_s = 0$.

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Disc entangling region

- Consider an entangling region which is a disc of radius R.

$$S_{\rm ren} = -\frac{\pi}{2G},$$

where G is dimensionless.

- This EE is related to the F quantity by the CHM map.

Casini-Huerta-Myers map

- Starting from

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2$$

let

$$t = R\frac{\cos\theta \sinh\tau/R}{(1+\cos\theta\cosh\tau/R)}$$

$$r = R\frac{\sin\theta}{(1+\cos\theta\cosh\tau/R)}$$

so

$$ds^2 = \Omega^2(-\cos^2\theta d\tau^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2))$$

- Covers $0 \leq r < R$ in original coordinates, i.e. disc.

Partition function and EE

- State in de Sitter is thermal with $\beta = 2\pi R$.
- Entanglement entropy for disc is mapped to thermodynamic entropy, which in turn is related to the partition function

$$S_{\rm deSitter} = -W = \ln Z$$

- Working in Euclidean signature the disc entanglement entropy is thus proportional to the partition function on S^3.

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Matching holographic renormalization schemes

- The counterterms for $AlAdS_4$ manifolds are (de Haro et al)

$$I_{\rm ct} = \frac{1}{8\pi G} \int d^3x \sqrt{h} (K + 2 - \frac{\mathcal{R}}{2})$$

  There are no possible finite counterterms.
- The renormalized onshell action for Euclidean AdS_4 gives

$$F = \frac{\pi}{2G}.$$

Scheme dependence

- The holographically renormalized partition function F indeed matches our renormalized disc entanglement entropy.
- Note that there are no possible finite terms in either renormalization scheme.

Outline

- **Renormalized entanglement entropy**
  1. CFT case: minimal surfaces in AdS
  2. **General definition**
- The F theorem

Definition of REE

- The holographic area renormalization of minimal surfaces is hard to connect with field theory renormalization.
- Related to the issue of first principles derivation of the holographic entanglement entropy formula (cf Lewkowycz/Maldacena).

General definition of renormalized entanglement entropy

- Entanglement entropy is often computed using the replica trick:

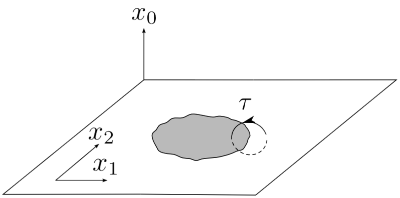
$$S = -n\partial_n \left[\log Z(n) - n \log Z(1)\right]_{n=1}$$

  where $Z(1)$ is the usual partition function and $Z(n)$ is the partition function on the replica space in which a circle coordinate has periodicity $2\pi n$.

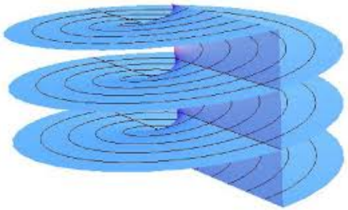
- Formally, to renormalise the entanglement entropy we can work with renormalized partition functions

$$S_{\rm ren} = -n\partial_n \left[\log Z_{\rm ren}(n) - n \log Z_{\rm ren}(1)\right]_{n=1}$$

Replica trick



3d field theory: on replica space τ has periodicity $2\pi n$.



Visualisation of $n = 3$ replica space.

Holographic renormalization

- Holographically $\log Z_{\rm ren}(n)$ is computed by the renormalised onshell action for a geometry with a conical singularity (Lewkowycz and Maldacena).
- From the replica formula, we can then derive the Ryu-Takayanagi formula.
- The standard holographic renormalization counterterms for the bulk action then imply our counterterms!

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Holographic renormalization

- For example, for an asymptotically locally AdS_4 spacetime the action counterterms are

$$I_{\rm ct} = \frac{1}{8\pi G} \int d^3x \sqrt{h} (K + 2 - \frac{\mathcal{R}}{2})$$

- For the replica space (Solodukhin)

$$\mathcal{R}_n = \mathcal{R} + 4\pi(n-1)\delta_\Sigma + \mathcal{O}(n-1)^2$$

where Σ is the conical singularity, i.e. the boundary of the entangling surface.

Holographic renormalization

- Applying the replica formula then leads to exactly

$$S_{\rm ct} = -\frac{1}{4G}\int dx \sqrt{\gamma}$$

  i.e. our counterterm localized on the boundary of the entangling surface.
- Procedure works for higher derivative theories such as Gauss-Bonnet.
- We can match renormalization scheme for EE with that of action!

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The F theorem

Weak version:

- $F_{UV} > F_{IR}$.

Strong version:

- F decreases monotonically under any relevant deformation.

To test the latter, we need to consider RG flows.

Holographic RG flows

A holographic RG flow is described by:

- A "domain wall" geometry

$$ds^2 = dr^2 + \exp(2A(r))dx^\mu dx_\mu$$

- A set of scalar field profiles

$$\phi_a(r)$$

- First order equations of motion relating $A(r)$ and $\phi_a(r)$.

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Renormalized EE

- The bare Ryu-Takayanagi EE depends only on the Einstein metric

$$S = \frac{1}{4G}\int d^{d-1}x\sqrt{\gamma}$$

- The counterterms can and do depend on the matter:

$$S_{\rm ct} \sim \int d^{d-2}x\sqrt{h}\mathcal{L}(\mathcal{R},\mathcal{K},\phi_a,\nabla\phi_a,\cdots)$$

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RG flow of 3d field theory

Consider four dimensional bulk ($d = 3$), single scalar ϕ.

- Assume UV conformal, so potential can be expanded near boundary as

$$V = 6 - \sum_{n=2}^{\infty} \frac{\lambda_{(n)}}{n!} \phi^n$$

with $\lambda_{(2)} = M^2 = \Delta(\Delta - 3)$.

- First order form of equations

$$\dot{A} = W \qquad \dot{\phi} = -2\partial_\phi W$$

where V is a known expression quadratic in (fake) superpotential W.

REE for relevant deformations

- We need the following counterterms in the REE:

$$S_{\rm ct} = -\frac{1}{4G}\int dx\sqrt{h}\left(1 + \frac{(3-\Delta)}{8(5-2\Delta)}\phi^2 + \cdots\right)$$

  where second term is needed for $\Delta > 5/2$.
- The counterterms can be expressed in terms of the superpotential

$$S_{\rm ct} = -\frac{1}{4G}\int dx\sqrt{h}Y(\phi)$$

  where

$$W(\phi)Y(\phi) + \frac{dW}{d\phi}\frac{dY}{d\phi} = 1.$$

Anomalies

- For $\Delta = 5/2$ the counterterm becomes logarithmic, i.e.

$$S_{\rm ct} = -\frac{1}{128G} \int dx \sqrt{h_E} \phi^2 r_0$$

  where r_0 is the UV cutoff.
- In general, conformal anomalies already known to arise for $\Delta = d/2 + 1$, see (Rosenhaus and Smolkin; Jones and MT)
- Additional conformal anomalies ($\sim \phi^n$) arise at $\Delta = 3 - 1/n$ ($\Delta = d + (2-d)/n$).

F quantity

- Under a relevant deformation of the 3d CFT

$$I \to I + \int d^3x \, \lambda \mathcal{O}_\Delta$$

the F quantity should decrease:

$$F(\lambda) = F_{UV} - \lambda^2 F_{(2)} + \mathcal{O}(\lambda^3)$$

with $F_{(2)} > 0$.

Renormalized entanglement entropy

- Compute the change in the REE to order λ^2, using the change in the Einstein metric induced.
- For $\Delta > 3/2$

$$\delta S_{\text{ren}} = \frac{\pi}{16(2\Delta - 5)G}\lambda^2 R^{2(3-\Delta)} + \mathcal{O}(\lambda^3).$$

  Positive for $\Delta > 3/2$ (agrees with F theorem) but negative for $\Delta < 5/2$!
- Note that for $\Delta = 3$:

$$\delta S_{\text{ren}} = 0$$

Change of F quantity under RG flows

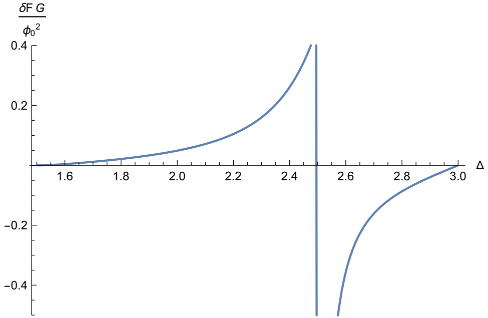
- We can also compute the change in F directly, by computing the onshell action for a curved domain wall solution

$$ds^2 = dw^2 + e^{2A(w)} ds^2_{S^3} \qquad \phi(w)$$

corresponding to the RG flow on S^3.

- Surprisingly this more involved (numerical) calculation has never been done.

Change of F quantity under RG flows



- δF is positive for $\frac{3}{2} \leq \Delta_+ < \frac{5}{2}$: strong F theorem is false!
- In our flows $F_{IR} < F_{UV}$.

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Dual quantizations

- For $-d^2 \leq 4m^2 \leq -d^2 + 4$, two quantisations are possible, $\Delta_{\pm}$.
- If for the Δ_+ operator

$$F = F_{(0)} + F_{(2)}\phi_{(0)}^2 + \cdots$$

with $\phi_{(0)}$ the source then for the Δ_- operator

$$F = F_{(0)} - \frac{1}{4F_{(2)}}\Psi_{(0)}^2 \cdots$$

where $\Psi_{(0)}$ is the source.

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Violation of the strong F theorem

- Either Δ_+ quantization or the Δ_- quantization must violate the strong F theorem!
- Both quantizations arise in holographic theories such as ABJM.
- Strong F theorem does not hold.... unless our models are in the swampland.

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Conclusions and outlook

Renormalized entanglement entropy is useful in field theory applications of entanglement entropy.

To explore further:

- Applications of REE to phase transitions, first law etc.
- Field theory calculation of renormalized EE using the replica trick (i.e. partition functions).

Conclusions and outlook

The strong F theorem does not seem to hold.

However:

- Does the weaker version $F_{UV} \geq F_{IR}$ always hold?
- Is there a modified version of the strong F theorem?